Finishing Cross Products

Recall we had

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}.$$

This has a few interesting effects. First $\mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}$. Also, it calculates vectors orthogonal to the originals:

$$(\mathbf{x}\times\mathbf{y})\cdot\mathbf{x} \;=\; (\mathbf{x}\times\mathbf{y})\cdot\mathbf{y} \;=\; 0.$$

There are some nice, basic, properties

$$(a\mathbf{x})\times\mathbf{y} = a(\mathbf{x}\times\mathbf{y}) \quad , \quad \mathbf{0}\times\mathbf{x} = \mathbf{0} \quad , \quad \mathbf{x}\times\mathbf{x} = \mathbf{0}$$

$$(\mathbf{x}+\mathbf{y})\times\mathbf{w} = \mathbf{x}\times\mathbf{w} + \mathbf{y}\times\mathbf{w}$$

Finally

$$\|\mathbf{x}\times\mathbf{y}\|^2 + (\mathbf{x}\cdot\mathbf{y})^2 = (\|\mathbf{x}\|\;\|\mathbf{y}\|)^2\,.$$

All of these can be proven with a little effort.

Some Geometry

The relevant section in the text is 3.3, titled 'Lines and Planes'

Points

A 'point' is simply that, a location in an $\mathbb{R}^n$ space, generally written $X = (x_1, x_2, x_3, \ldots, x_n)$ if we want to be clear it's a point, or in a more vector type form. So, it's one of the following three

$$(x_1\,,\,x_2\,,\,\cdots\,,\,x_n)\,, \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{or} \qquad [x_1 \quad x_2 \quad \ldots \quad x_n]\,.$$

A vector in these spaces can be described as the displacement between two points, so

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \overrightarrow{OX}, \qquad \text{using} \quad O = (0,0), \text{ and } X = (1,2).$$

That expression $\overrightarrow{OX}$ means displacement (the transition, the move, etc), from O to X. The order is important: $\overrightarrow{OX} = -\overrightarrow{XO}$.

Every single vector $\mathbf{x}$ is equal to $\overrightarrow{OX}$ where X is a point with the exact same values as $\mathbf{x}$. As a result, there is very little distinction to be made between these, especially mechanically.

One key thing to remember is that you *cannot add points* together. You can, however, add vectors, as we have seen.

Lines

Back in calculus, we would use lines of the form

$$y = ax + b, \qquad a = \text{ slope}, \qquad b = \text{height at } x = 0.$$

This misses out on some options, it's better to forget the 'y dependent on x' setup and use

$$ay + bx = c$$

which allows, for instance, $x = 2$, a vertical line.

Thing is, we are in $\mathbb{R}^n$, not just $\mathbb{R}^2$, so ours will be of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} r + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad \mathbf{x} = \mathbf{a}\, r + \mathbf{b}.$$

This is the same thing, really, just more general, called the 'parametric form' of the line.

Example: Here are two simple lines

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} r, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} s.$$

They are actually the same line. To confirm that there are two procedures we can take, first, we can simply equate the lines, confirm that the set of points is the same in each line. Here's how that looks:

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} r \quad = \quad \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} s$$

which becomes

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} s - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} r.$$

This is then a two by three system, just solve it. Easy enough. Here another way.

Every line can be *fully* characterized by a *single* point and a single vector (direction). To find if two lines are the same, make sure they have the same direction to them, the vectors should be simple multiples of each other. This is obvious here, since

$$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

They then have the same shape, are parallel, etc. Now we just need to get a single point in common. For instance,

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}, \qquad \text{so, } s = -1.$$

Intersections of Lines:

Checking intersection can be lengthy, but the principle is fairly simple. Just equate the lines, so if we have

$$\mathbf{x} = \mathbf{a} + \mathbf{b}t \qquad , \qquad \mathbf{y} = \mathbf{c} + \mathbf{d}s$$

then we want to find if there are t, s such that

$$\mathbf{x} = \mathbf{y} \quad \Longrightarrow \quad \mathbf{a} + \mathbf{b}t = \mathbf{c} + \mathbf{d}s$$

which works out to the system

$$\mathbf{b}t - \mathbf{d}s = \mathbf{c} - \mathbf{a}.$$

Example: Do the following lines intersect:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

intersect is simply about equating $\mathbf{x} = \mathbf{y}$ which becomes

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} s - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t.$$

We get the set of equations

$$\begin{aligned} -1 = -t \Longrightarrow & \quad t = 1 \\ 1 = -s \Longrightarrow & \quad s = -1 \\ 0 = s - t & \end{aligned}$$

and if you check the last row, they do not match, so the lines do not intersect.

There are three different types of intersection:

-An infinite number of points of intersection, like the first example, which means the lines are identical.

-A single point of intersection, meaning they cross each other.

-No intersection, like the second example.

Note that two lines in $\mathbb{R}^2$ that are not parallel will intersect. Once you are in $\mathbb{R}^3$, they can not intersect without being parallel.

Planes

We will stick to $\mathbb{R}^3$ for a while here, since planes have a particularly nice property in $\mathbb{R}^3$.

So, planes: they have height, width, but no depth. They are FLAT. If you are on part of one, at, say, the point $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ then you can move in 2 distinct directions, so we get something like

$$\mathbf{x} = \mathbf{a} + \mathbf{b}t + \mathbf{c}s \qquad \Longrightarrow \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} s, \quad t, s \in \mathbb{R}.$$

That's the *parametric form* of a plane.

Next: an additional way to write a plane. As before, we are on a plane, at the point $\mathbf{a}$. We have the usual two directions to move, $\mathbf{b}$ and $\mathbf{c}$. We are in $\mathbb{R}^3$, so there has to be a third direction. We want to chose the *exact* wrong direction to stay in the plane, the direction *exactly* away from the plane, which has to be orthogonal to $\mathbf{b}$ and $\mathbf{c}$. We call this direction the *normal* of the plane, usually written $\mathbf{n}$.

Let us consider what happens when we dot product both sides

$$\begin{aligned} \mathbf{x} \cdot \mathbf{n} &= (\mathbf{a} + \mathbf{b}t + \mathbf{c}s) \cdot \mathbf{n} \\ &= \mathbf{a} \cdot \mathbf{n} + \mathbf{b} \cdot \mathbf{n}t + \mathbf{c} \cdot \mathbf{n}s \quad = \mathbf{a} \cdot \mathbf{n} \end{aligned}$$

which combines nicely for

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

Expanding that out with the column vector notation leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \quad n_1 x_1 + n_2 x_2 + n_3 x_3 \quad = \mathbf{a} \cdot \mathbf{n}.$$

Note that the right hand side is simply a real number, as $\mathbf{n}$ and $\mathbf{a}$ are just constant vectors.

This is frequently written

$$ax + by + cz = d$$

or something similar, depending on what variables we are using. As long as a, b and c are *not all* equation to zero, this gives us a plane in $\mathbb{R}^3$. The semi-official name for this expression: the *scalar equation* of a plane.

Example: Convert the scalar equation $4x - 2y + z = 2$ into a parametric form.

This is not a big deal, usually. We have the coefficients, 4, -2 and 1, so the normal is $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$. There are three values in $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$, it's not $= \mathbf{0}$, so there will be 2 vectors orthogonal to it. Any two vectors that are not in the same direction will do. The best way to make sure they aren't in the same direction is to make them cover different variables, i.e., use only x and y for one, y and z for the other.

For instance, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$, as is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. Those are clearly not in the same direction. They'll do. Recall that we need to match the right hand side. *Any* combination of those two will not change the total on the left. We just need a single point that matches. So,

$$x = \frac{1}{2} \Longrightarrow \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \qquad y = -1 \Longrightarrow \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \qquad z = 2 \Longrightarrow \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

So, any will do, here's one final set

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} s.$$

Example: Convert the scalar equation $x - 2z = 0$ into a parametric form. We only need the orthogonal vectors, since we have a zero on the right. There is *no y term*, so we can use $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as an orthogonal vector. The remaining term is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Final answer

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} s.$$

Note: To get the normal vector for the equation, just use the cross product on the two vectors in the parametric form, then get a starting point. As usual, *any* point output by the parametric equation will do.

Example: Convert

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} s$$

into the scalar equation form.

First, we use the cross product to calculate a normal vector

$$\mathbf{n} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -6-2 \\ -3+1 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -2 \end{bmatrix}.$$

This give us an equation of the form

$$4x_1 - 8x_2 - 2x_3 = \text{ something}$$

so we need to find a right hand side value.

As usual, we just need to make it match at one point. If we use $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, the left hand side will be

$$4(1) - 8(0) - 2(0) = 2$$

so our final answer is

$$4x_1 - 8x_2 - 2x_3 = 2.$$

Intersections of Planes

These can be a lot of work. If you are using the parametric equations, it is like solving for intersections of lines, except there are more variables to worry about. It is more likely we will do this using the scalar equations.

Example: find the intersections of $2x - 3y + z = 2$ and $4x - z = -1$.

Basic idea: assume that the x, y, z in the one equation is equal to the x, y, z in the other then see what that tells you.

First, we can re-arrange the second equation for $z = 4x + 1$. Next, we substitute that into the other equation for

$$\begin{aligned} 2x - 3y + (4x + 1) &= 2 \\ 6x - 3y &= 1 \\ 3y &= 6x - 1 \\ y &= 2x - \frac{1}{3} \end{aligned}$$

and both y, z are written in terms of x so our answer is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2x - \frac{1}{3} \\ 4x + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}.$$

We get a line. This is one of the three options available:

-No intersection (so they are parallel and disjoint).

-A line (they intersect, but are not parallel).

-Full intersection (they are identical).

Compare with the available answers for two lines in $\mathbb{R}^2$, where any line can be written

$$ax + by = c.$$

Now for a few questions. The text uses a slightly different format some times, called the scalar equation of a line:

$$\begin{array}{rcl} x &=& 3 + t \\ y &=& 1 - t \\ z &=& -3 + 2t \end{array} \quad \text{equivalent to} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} t$$

4.b), this may require some thought.

7.b),d)

9. (note: no answer at the back)

10.b)d)

12.b),d)

17. (also no answer at the back)

18.